

The quantum relativistic harmonic oscillator: generalized Hermite polynomials \star

V. Aldaya ^a, J. Bisquert ^{b,c} and J. Navarro-Salas ^b

^a *Departamento de Física Teórica y del Cosmos, Facultad de Ciencias, Universidad de Granada, Granada, Spain*

^b *Departamento de Física Teórica, Facultad de Física, Universidad de Valencia, Burjasot, 46100 Valencia, Spain and IFIC (Centro mixto Universidad de Valencia-C.S.I.C), Burjasot, 46100 Valencia, Spain*

^c *Departamento de Física, Escuela Universitaria Politécnica, Universidad de Castilla - La Mancha, 02071 Albacete, Spain*

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A relativistic generalisation of the algebra of quantum operators for the harmonic oscillator is proposed. The wave functions are worked out explicitly in configuration space. Both the operator algebra and the wave functions have the appropriate $c \rightarrow \infty$ limit. This quantum dynamics involves an extra quantization condition $mc^2/\omega\hbar = 1, \frac{3}{2}, 2, \dots$ of a topological character.

It is well known that great difficulties arise when generalizing any quantum equation for bound states to the relativistic case. In particular, no proper relativistic quantum equation for the harmonic oscillator seems to have been found, notwithstanding the simplicity of the Galilean counterpart. This sort of relativistic system is of general applicability in physics as a first approximation to periodic dynamics with positive energy. A case in which this relativistic dynamics might acquire a major significance is in the quantum-mechanical construction of hadrons in terms of quarks. Although the best models of hadrons are based on a non-relativistic, harmonic oscillator potential, the fact that the velocities of quarks are not small compared with c suggests that relativistic corrections should be taken into account [1]. In this context, another proposed approach makes use of a naive Lorentz-covariant generalization $x^\mu x_\mu$ of the harmonic oscillator potential thus leading to wave functions having dependence on time analogous to the space one, i.e. a Gaussian function times the ordinary Hermite polynomials [2,3]. Also in the theory of quantum dynamics on curved space-time a

quantum relativistic harmonic oscillator (RHO) should be of special interest [4].

In this Letter we tackle the task of finding a relativistic algebra (the simplest one, indeed) of quantum operators generalizing the non-relativistic one generated by the energy, position and momentum operators, i.e.

$$[\hat{E}, \hat{x}] = -i \frac{\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = im\omega^2 \hbar \hat{x},$$

$$[\hat{x}, \hat{p}] = i\hbar. \quad (1)$$

The new algebra must reproduce the algebra of the quantum free relativistic particle for $\omega \rightarrow 0$ as well as the algebra (1) in the $c \rightarrow \infty$ limit. Also in this limit the wave functions should recover those of the non-relativistic harmonic oscillator (NRHO), that is to say, the ground state must tend to a Gaussian in x and the excited ones to a Hermite polynomial times the Gaussian.

We propose the following algebra for the quantum operators associated with the RHO,

$$[\hat{E}, \hat{x}] = -i \frac{\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = im\omega^2 \hbar \hat{x},$$

$$[\hat{x}, \hat{p}] = i\hbar \left(1 + \frac{1}{mc^2} \hat{E} \right). \quad (2)$$

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The algebra (2) is an affine version of the $sO(2, 1) \approx sL(2, \mathbb{R})$ algebra, which is in turn the relativistic anti-de Sitter algebra in 1+1 dimensions [5] containing the Lorentz algebra in 1+1 dimensions $sO(1, 1)$ (the generalization to the 3+1 case, $sO(2, 3)$, which includes the rotational symmetry, is straightforward), and satisfies the above-mentioned requirements as can be trivially verified. In fact, the $\omega \rightarrow 0$ limit of (2) is the Poincaré algebra in 1+1 dimensions when \hat{E} has the rest-mass energy subtracted. The algebra (2) itself establishes the infinitesimal transformation properties of each one of the generators and, in particular, under the "boosts" here generated by \hat{x} (in the limit $\omega \rightarrow 0$, \hat{x} generates the ordinary Lorentz boosts in the same way as in the limit $\omega \rightarrow 0$, for the non-relativistic algebra (1), \hat{x} generates the Galilean boosts). The corresponding quantum dynamic system will be described by means of an irreducible representation (highest weight) of (2). \hat{E} , \hat{p} and \hat{x} will be differential operators acting on a Hilbert space of functions generalising the Hermite polynomials H_n endowed with a scalar product weighted with a certain localized function also generalizing $\exp(-m\omega x^2/2\hbar)$. The main difficulty lies in finding the representation in configuration space. Although the Bargmann-Fock-Segal-like (BFS) realization for the RHO has been already given in ref. [6], exhibiting the appropriate non-relativistic limit, it is not easy to find a Bargmann-like transformation connecting the configuration and Fock realizations. Therefore we shall seek the wave functions on configuration space directly from (2). As an additional demonstration of the relativistic character of our model we will explicitly show how these wave functions obey the quantum "Klein-Gordon" equation on a (negative constant curvature) anti-de Sitter universe. This fact establishes the mathematical equivalence between both physical systems.

The technique we follow for obtaining the quantum states consists in writing down an expression in (t, x, p) -space for the operators of the algebra (2) as first-order differential operators. This representation of functions $\psi(t, x, p)$ will be reducible. In fact, we shall show non-trivial operators commuting with \hat{x} , \hat{p} and \hat{E} . The reduction is achieved by trivializing the action of those operators commuting with the representation. Let us illustrate the mechanism with the well known case of the non-relativistic algebra

(1). A more general development can be found in ref. [7] (see also ref. [8]).

A set of first-order differential operators fulfilling (1) is given by

$$\begin{aligned} \frac{1}{i\hbar} \hat{x} &= \cos \omega t \frac{\partial}{\partial p} + \frac{\sin \omega t}{m\omega} \frac{\partial}{\partial x} \\ &\quad - \frac{i}{2\hbar} \left(x \cos \omega t - \frac{p}{m\omega} \sin \omega t \right), \\ -\frac{1}{i\hbar} \hat{p} &= \cos \omega t \frac{\partial}{\partial x} - \omega m \sin \omega t \frac{\partial}{\partial p} \\ &\quad + \frac{i}{2\hbar} (p \cos \omega t + \omega m x \sin \omega t), \\ \frac{1}{i\hbar} \hat{E} &= \frac{\partial}{\partial t}. \end{aligned} \quad (3)$$

The following differential operators,

$$\begin{aligned} L_p &\equiv m \frac{\partial}{\partial p} + \frac{imx}{2\hbar}, \quad L_x \equiv \frac{\partial}{\partial x} - \frac{ip}{2\hbar}, \\ L_t &\equiv \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - \omega^2 m x \frac{\partial}{\partial p}, \end{aligned} \quad (4)$$

commute with those in (3) and thus allow us to reduce the representation given by the action of (3) on functions ψ , which depend on x , p and t by imposing restrictions on the arguments. For instance, the maximum set of compatible conditions,

$$L_t \psi = 0 = \left(\sqrt{m\omega/2} L_p + \frac{i}{\sqrt{2m\omega}} L_x \right) \psi,$$

leads to the wave functions $\psi_n = \exp(-|z|^2)(z^\dagger)^n$, depending on the complex coordinate

$$z = \sqrt{m\omega/2\hbar} x + ip/\sqrt{2m\omega\hbar},$$

i.e. the irreducible representation in the BFS-representation. In order to obtain a realization in configuration space, on the other hand, one should impose the condition $L_p \psi = 0$, which nevertheless prevents the imposition of any other condition and the reduction is incomplete. That condition implies that $\psi = \exp(-ipx/2\hbar)\phi(t, x)$ so that the p -dependence factors out. The full reduction can be accomplished by choosing a vacuum, $|0\rangle \equiv \psi_0$, annihilated by \hat{E} and $\hat{z} \equiv \sqrt{m\omega/2\hbar} \hat{x} + i\sqrt{2m\omega\hbar} \hat{p}$.

The remaining states and hence the Hermite poly-

nomials are obtained by means of the action of the creation operator \hat{z}^\dagger on the vacuum. They are

$$\psi_n = \exp(-ipx/2\hbar) 2^{-n/2} \exp\left(-\frac{m\omega}{2\hbar} x^2\right) \\ \times \exp(-in\omega t) H_n(\sqrt{m\omega/\hbar} x).$$

Going back to the RHO we now write the generalization of the operators (3) closing the algebra (2), leading to the non-relativistic limit as well as the $\omega \rightarrow 0$ limit,

$$\frac{1}{i\hbar} \hat{x} = \left(\frac{P^0}{m\alpha} \cos \omega t + \frac{\omega xp}{mc^2 \alpha} \sin \omega t \right) \frac{\partial}{\partial p} \\ + \frac{\alpha}{m\omega} \sin \omega t \frac{\partial}{\partial x} + \frac{x}{mc^2 \alpha} \cos \omega t \frac{\partial}{\partial t} \\ - \frac{i}{\hbar \alpha (P^0 + mc)} \left(P^0 x \cos \omega t - \frac{P^0}{\omega} \sin \omega t \right), \\ - \frac{1}{i\hbar} \hat{p} = \alpha \cos \omega t \frac{\partial}{\partial x} - \left(\frac{P^0 \omega}{c\alpha} \sin \omega t \right. \\ \left. - \frac{\omega^2 xp}{c^2 \alpha} \cos \omega t \right) \frac{\partial}{\partial p} - \frac{\omega x}{c^2 \alpha} \sin \omega t \frac{\partial}{\partial t} \\ + \frac{im}{\hbar \alpha (P^0 + mc)} (p \cos \omega t + P^0 \omega x \sin \omega t), \\ \frac{1}{i\hbar} \hat{E} = \frac{\partial}{\partial t}, \quad (5)$$

where $\alpha = \sqrt{1 + \omega^2/c^2}$ and

$$P^0 = \sqrt{m^2 c^2 + p^2 + m^2 \omega^2 x^2}$$

(this definition of P^0 would be considered as the mass-shell constraint if we had introduced the extra variable P^0 as the canonically conjugate of ct).

Furthermore, the analogues of the operators (4) in the relativistic case, i.e. operators which commute with those of (5), are

$$L_p = \frac{P^0}{c} \frac{\partial}{\partial p} + \frac{im^2 cx}{\hbar (P^0 + mc)}, \\ L_x = \frac{P^0}{mc} \frac{\partial}{\partial x} + \frac{p}{mc^2 \alpha^2} \frac{\partial}{\partial t} - \frac{imcp}{\hbar (P^0 + mc)}, \\ L_t = \frac{P^0}{m\alpha^2} \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - m\omega^2 x \frac{\partial}{\partial p}. \quad (6)$$

We shall now make use of the procedure just ex-

plained for the NRHO. In order to obtain the wave functions in configuration space we have to impose the condition $L_p \psi = 0$, the general solution of which is

$$\psi(t, x, p) = \exp[if(t, x, p)/\hbar] \phi(t, x),$$

where

$$f(t, x, p) = -\frac{2mc^2}{\omega} \arctan\left(mc^2(\alpha - 1) \frac{P^0/mc - \alpha}{\omega px} \right)$$

is a common factor analogous to the non-relativistic one, $\exp(-ipx/2\hbar)$, also factorizing the p -dependence of ψ .

As in the non-relativistic case, the vacuum, ψ_0 , will be a function verifying $L_p \psi_0 = 0$ and will be annihilated by the operators \hat{E} and

$$\hat{z} \equiv \sqrt{m\omega/2\hbar} \hat{x} + i\hat{p}/\sqrt{2m\omega\hbar},$$

$\psi_0 \equiv \exp[if/\hbar] \Phi_0(x)$, where $\Phi_0(x) = \alpha^{-N}$ and N is the dimensionless number $mc^2/\hbar\omega$. Observe now that the new "weight function" $\Phi_0(x)$ reduces to the Gaussian one when $c \rightarrow \infty$. The excited states, ψ_n , can be found by applying the creation operator \hat{z}^\dagger to the ground state:

$$\psi_n = (\hat{z}^\dagger)^n \psi_0 \\ = \exp(if/\hbar) \exp(-in\omega t) 2^{-n/2} \Phi_n^N(x) \\ = \exp(if/\hbar) \exp(-in\omega t) 2^{-n/2} \alpha^{-(N+n)} H_n^N(\xi),$$

where $H_n^N(\xi)$ are polynomials, which we shall call the relativistic Hermite polynomials (RHP), and $\xi \equiv \sqrt{m\omega/\hbar} x$.

Let us look at the dimensionless number, $N = mc^2/\hbar\omega$, introduced above. As far as the quantum algebra is concerned a restriction to N comes from the square integrability of wave functions, which implies $N > \frac{1}{2}$ without any non-relativistic analogue. Moreover, if we desire our wave functions to support the irreducible representations (highest weight) of the $SL(2, \mathbb{R})$ group, rather than those of the Lie algebra only or the universal covering group of $SL(2, \mathbb{R})$, a quantization condition appears for N , which must be a half-integer. This number plays exactly the same role as the spin in the $SU(2)$ representations and characterizes the unitary, highest weight, irreducible representations of $SL(2, \mathbb{R})$.

The expression $\psi_n = (\hat{z}^\dagger)^n \psi_0$ along with the Casi-

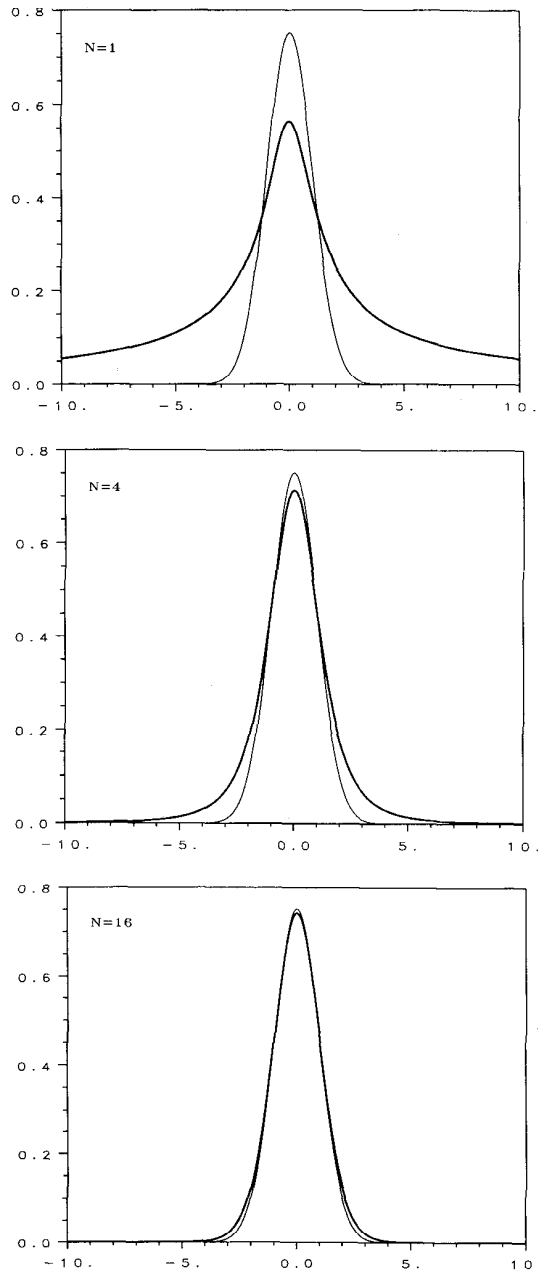


Fig. 1.

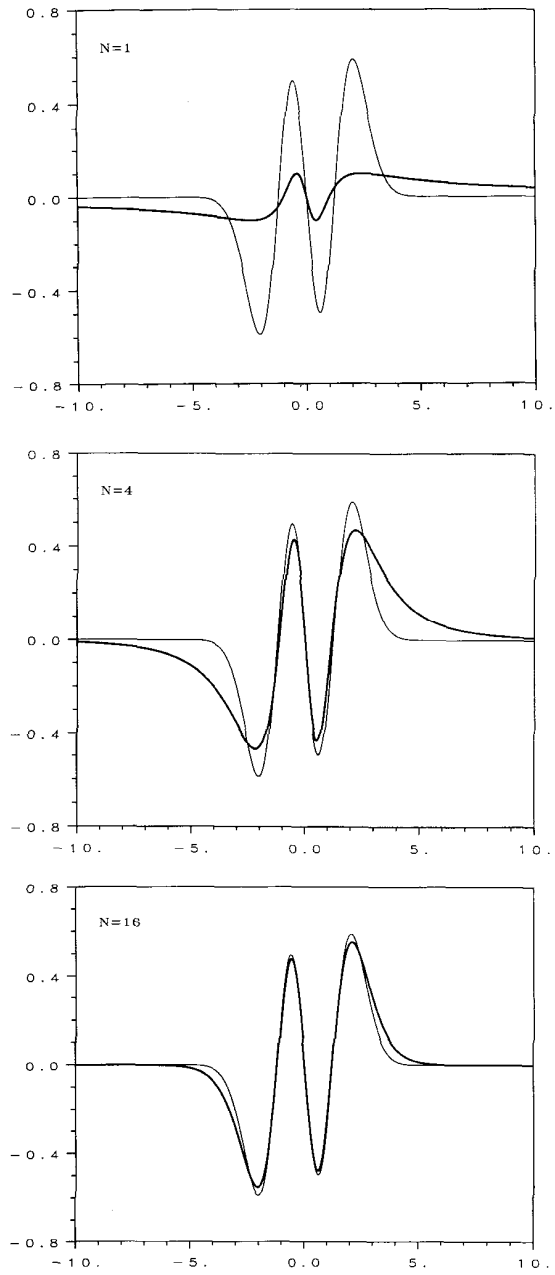


Fig. 2.

mir operator of the representation provides the recurrence relation among the $H_n^N(\xi)$ as well as the second-order differential equation for them. More precisely

$$H_{n+1}^N(\xi) = \left[2\left(1 + \frac{n}{N}\right)\xi - \left(1 + \frac{\xi^2}{N}\right) \frac{d}{d\xi} \right] H_n^N(\xi), \quad (7)$$

$$\frac{d}{d\xi} H_n^N(\xi) = \frac{n}{N} (2N+n-1) H_{n-1}^N(\xi), \quad (8)$$

$$\left(1 + \frac{\xi^2}{N}\right) \frac{d^2}{d\xi^2} H_n^N - \frac{2}{N} (N+n-1) \xi \frac{d}{d\xi} H_n^N + \frac{n}{N} (2N+n-1) H_n^N = 0. \quad (9)$$

From (7)–(9) we can work out the general expression for the RHP,

$$H_n^N(\xi) = \sum_{s=0}^{\lfloor n/2 \rfloor} a_{n,n-2s}^N (2\xi)^{n-2s}, \quad (10a)$$

$$a_{n,n-2s} = \frac{(-1)^s n! N^s (N-\frac{1}{2})! (2N+n-1)!}{s! (n-2s)! (N+s-\frac{1}{2})! (2N)^n (2N-1)!}. \quad (10b)$$

The first polynomials are

$$H_0^N = 1, \quad H_1^N = 2\xi, \quad H_2^N = 4\left(1 + \frac{1}{2N}\right)\xi^2 - 2, \\ H_3^N = 8\left(1 + \frac{3}{2N} + \frac{1}{2N^2}\right)\xi^3 - 12\left(1 + \frac{1}{N}\right)\xi. \quad (11)$$

It is easy to see that at the limit $c \rightarrow \infty$ ($N \rightarrow \infty$) all the expressions above go to the non-relativistic counterparts. Moreover, it is not difficult to test from (9) that the wave functions $\tilde{\psi}_n \equiv \exp(-in\omega t) \Phi_n^N(x)$ satisfy the Klein–Gordon equation associated with the anti-de Sitter metric, $ds^2 = c^2 \alpha^2 dt^2 - \alpha^{-2} dx^2$,

$$(\square + m^2 c^2 / \hbar^2 + \frac{1}{2} NR) \tilde{\psi}_n = 0,$$

where $R = -2\omega^2/c^2$ is the curvature. For the sake of completeness we give the normalization of the wave functions,

$$\langle \Phi_n^N, \Phi_m^N \rangle = \int dx \alpha^{-(2N+n+m)} H_n^N H_m^N \\ = 2^n n! \frac{(2N+n-1)!}{(2N-1)!} \frac{1}{(2N)^n} \frac{(2N-3)!!}{(2N-2)!!} \sqrt{N} \pi \delta_{nm}. \quad (12)$$

In figs. 1 and 2 we have plotted (thick line) the normalized wave functions $\Phi_0^N(\xi)$ and $\Phi_3^N(\xi)$, respectively, for three different values of N . The corresponding non-relativistic wave functions have also been plotted (thin line) for comparison.

We finally point out the fact that the spacing of the energy levels is identical to the non-relativistic one and the shape of the wave functions is appreciably different for low N , as shown in figs. 1 and 2. In particular, for $N=1$ the ratio between the values at the origin of the relativistic and non-relativistic ground state is about 22%. We believe that these two facts may well account for the unexpected success of the non-relativistic, harmonic-oscillator-based models for hadron spectroscopy, as well as the need for a lowering of the ground-state wave function [9].

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